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HYBRID EXPANSION METHOD FOR FREQUENCY RESPONSES AND THEIR SENSITIVITIES, PART II: VISCOUSLY DAMPED SYSTEMS

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Frequency responses and their sensitivities have been broadly applied to finite element model updating, structural damage detection, dynamic optimization, vibration control and so on. In this paper, the modal acceleration method for the frequency responses and the double-modal acceleration method for their sensitivities, which have been discussed in the previous paper for undamped systems, are extended to viscously damped systems. The two methods are based on the hybrid expansion, power series expansion and modal superposition, of the inverse of the complex dynamic system matrix. Three steps are required to calculate the sensitivities using the proposed method. Firstly, frequency responses of a system excited by external forces are calculated by using the modal acceleration. Pseudo-force vector is then computed from the production of the sensitivity matrix and the frequency response vector. Finally, a second modal acceleration is applied to obtain the general frequency responses under the pseudo-forces, that is, the sensitivities. Two modal truncation schemes, middle-high-modal and low-high-modal truncation schemes, are presented according to the values of the exciting frequencies. The modal truncation errors of the modal acceleration method for frequency responses and the double-modal acceleration method for the sensitivities are also given to show the convergence of the proposed methods. Although the frequency responses and their sensitivities are discussed in this paper, the proposed methods are also valid for the frequency response functions, responses in time domain and their sensitivities. The results of a floating raft isolation system show that the proposed modal acceleration methods are efficient, especially for the sensitivity analysis. The modal truncation errors of the frequency responses and their sensitivities will reduce quickly when the two-modal acceleration methods are adopted.

1. INTRODUCTION

The sensitivities of dynamic properties with respect to selected structural parameters have been applied to finite element model updating, structural damage detection, dynamic optimization, vibration control and so on. The sensitivities of eigenvalues and their corresponding eigenvectors of a structure have been discussed in detail during the past 30 years. The results of such work are fruitful and almost conclusive. However, there seems to be far less work being done directly on the frequency responses, which have even more practical applications.

Generally, direct method and modal superposition method are used to calculate frequency responses. The former is based on the direct frequency solution and results in an exact calculation of frequency responses. The latter can be classified into several approaches: typical modal superposition [1], modified modal superposition [1] and modal

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acceleration approaches [2, 3]. Three kinds of methods, direct approach [4, 5], modal superposition approach [6, 7] and double-modal superposition approach, are usually used to calculate the sensitivities of frequency responses. In 1993, Ting [8] proposed an improved method for calculating the sensitivities of frequency responses. It combines the modal acceleration method with the Ritz minimization technique to improve the modal approximation accuracy. The advantages and disadvantages of these methods were reviewed in detail by Qu [9].

Recently, a modal acceleration method for frequency responses and a double-modal acceleration method for their sensitivities have been proposed by Qu [9]. It can be proved that only the modes lie within the interested frequency range are required to calculate the frequency responses and their sensitivities using these methods. Their accuracy increases quickly with the increase of the number of the item in power series.

Unfortunately, almost all the aforementioned approaches were proposed for undamped systems. Although some of them, in reference [2] for example, can be used to proportionally damped systems, they are not convergent for most cases. As one knows, the viscously damping is very important for some systems, especially for vibration isolation systems and active vibration control systems. The frequency responses and their sensitivities, other than eigen sensitivities, are much more useful for the dynamic design of such systems due to their better properties [10].

In this paper, the modal acceleration method and the double-modal acceleration method [9] are extended to viscously damped systems. The modal superposition method for frequency responses and the double-modal superposition method for their sensitivities are presented in section 3. According to the exciting frequency, two modal truncation schemes, middle–high-modal and low–high-modal, are proposed. The corresponding modal truncation errors of the frequency responses and their sensitivities are also given in this part.

Based on the power series expansion and modal superposition of the inverse of the complex dynamic system matrix, a modal acceleration method and a double-modal acceleration method for frequency responses and their sensitivities in the low-frequency range are derived in section 4. Only the modes whose corresponding frequencies lie in the range of the exciting frequencies are required to calculate the frequency responses and their sensitivities. The modal truncation errors are also provided in this part for comparison purpose.

Sometimes, the interested frequencies do not lie in the low-frequency range of a system or structure. Floating raft isolation system [10] is a good example. Usually, the vibration frequencies of the main and auxiliary machines in a ship or submarine lie in the range 1000–6000 rad/s. The frequencies of the structural vibration are higher than this. However, the lowest frequency of the isolation system, such as floating rafting isolation, is usually much smaller than these frequencies. This means the exciting frequencies lie in the middle frequency range of the isolation system. When we design or optimize this kind of isolation system, the vibration transmissibility, which is a function of the frequency responses, in that frequency range is required. If we use all the modes within the low- and middle-frequency range to calculate the frequency responses and their sensitivities, the number of the kept modes will be very large. This is a heavy burden for the eigensolution analysis. If only the modes which lie in the interested frequency range are selected, the modal truncation error is usually very large, especially for the sensitivities. Based on the eigenvalue shifting technique, the modal acceleration method for frequency responses and the double-modal acceleration method for their sensitivities are derived in section 5.

A floating raft isolation system for ships and submarines is included in section 6 to demonstrate the efficiency of the proposed methods. The results will show that the modal truncation errors of the frequency responses and their sensitivities reduce quickly when the two-modal acceleration methods are adopted.

2. THEORETICAL BACKGROUND

The dynamic equations of an n-degree-of-freedom (d.o.f.) viscously damped system in frequency domain can be written in matrix form as

$$\lfloor \mathbf{K}(\mathbf{p}) + \mathrm{j}\omega C(\mathbf{p}) - \omega^2 \mathbf{M}(\mathbf{p}) \rfloor \mathbf{X}(\mathbf{p}, \omega) = \mathbf{F}(\omega), \tag{1}$$

where $\mathbf{M}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$ and $\mathbf{K}(\mathbf{p})$ are mass, damping and stiffness matrices respectively. They are the functions of design parameter vector

$$\mathbf{p} = \{p_1, p_2, \ldots, p_m\}^{\mathrm{T}}. \tag{2}$$

For simplicity, this indication will be omitted in the following. $\mathbf{X}(\mathbf{p}, \omega)$ and $\mathbf{F}(\omega)$ are the displacement and exciting force vectors respectively. ω is an exciting frequency and $\mathrm{j} = \sqrt{-1}$. Because the viscous damping matrix is not proportional to the stiffness and/or mass matrices, the dynamic equations (1) are difficult to be uncoupled in physical space. It is necessary to introduce $2n$-dimensional state space, i.e.,

$$\mathbf{Y}(\omega) = \begin{Bmatrix} \mathbf{X}(\omega) \\ \mathrm{j}\omega \mathbf{X}(\omega) \end{Bmatrix}. \tag{3}$$

Hence, the dynamic equations (1) can be rewritten in the state space as

$$(\mathbf{A} - \mathrm{j}\omega \mathbf{B}) \mathbf{Y}(\omega) = \mathbf{F}_S(\omega), \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -\mathbf{C} & -\mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{F}_S(\omega) = \begin{Bmatrix} \mathbf{F}(\omega) \\ \mathbf{0} \end{Bmatrix}. \tag{5}$$

The frequency responses can be obtained from equation (4) as

$$\mathbf{Y}(\omega) = (\mathbf{A} - \mathrm{j}\omega \mathbf{B})^{-1} \mathbf{F}_S(\omega). \tag{6}$$

Equation (6) is the governing equation of frequency responses of viscously damped systems in state space.

The sensitivities of the frequency responses can be obtained by taking the first partial derivative of equation (4) with respect to a selected design variable $p_j (j = 1, 2, \ldots, m)$, that is,

$$(\mathbf{A} - \mathrm{j}\omega \mathbf{B}) \frac{\partial \mathbf{Y}(\omega)}{\partial \mathbf{p}_j} = \mathbf{R}(\omega), \tag{7}$$

where pseudo-force vector $\mathbf{R}(\omega)$ is defined as

$$\mathbf{R}(\omega) = \mathbf{S}(\omega) \mathbf{Y}(\omega) \tag{8}$$

and the sensitivity matrix is

$$\mathbf{S}(\omega) = -\frac{\partial \mathbf{A}}{\partial p_j} + \mathrm{j}\omega \frac{\partial \mathbf{B}}{\partial p_j}. \tag{9}$$

Hence, the sensitivities of frequency responses are

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = (\mathbf{A} - \mathrm{j}\omega\mathbf{B})^{-1}\mathbf{R}(\omega). \tag{10}$$

Equation (10) is the governing equation of the sensitivities of frequency responses in state space.

3. MODAL SUPERPOSITION METHOD AND DOUBLE-MODAL SUPERPOSITION METHOD

Assume that the eigenvalue and eigenvector matrices of the viscously damped system are $\tilde{\mathbf{\Omega}}$ and $\tilde{\mathbf{\Psi}}$. According to the theory of the complex modal, they can be expressed in submatrix form as

$$\tilde{\mathbf{\Psi}} = \begin{bmatrix} \mathbf{\Psi} & \mathbf{\Psi}^* \\ \mathbf{\Psi\Omega} & \mathbf{\Psi}^*\mathbf{\Omega}^* \end{bmatrix}, \qquad \tilde{\mathbf{\Omega}} = \begin{bmatrix} \mathbf{\Omega} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Omega}^* \end{bmatrix}, \tag{11}$$

where the superscript "*" denotes complex conjugation. The eigenvalue and eigenvector matrices should satisfy the following eigenequation and orthogonal conditions:

$$\mathbf{A}\tilde{\mathbf{\Psi}} = \mathbf{B}\tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}, \qquad \tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{A}\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Omega}}, \tag{12, 13}$$

$$\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{B}\tilde{\mathbf{\Psi}} = \mathbf{I}, \qquad \tilde{\mathbf{\Psi}}^{\mathrm{T}}(\mathbf{A} - \mathrm{j}\omega\mathbf{B})\tilde{\mathbf{\Psi}} = \tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I}, \tag{14, 15}$$

where $\mathbf{I}$ is an identity matrix of $2n \times 2n$. From equations (13) and (15) one obtains

$$\mathbf{A}^{-1} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}, \qquad (\mathbf{A} - \mathrm{j}\omega\mathbf{B})^{-1} = \tilde{\mathbf{\Psi}}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}. \tag{16, 17}$$

3.1. MODAL SUPERPOSITION METHOD FOR FREQUENCY RESPONSES

Introducing equation (17) into equation (6), one has

$$\mathbf{Y}(\omega) = \tilde{\mathbf{\Psi}}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{F}_S(\omega), \tag{18}$$

Assuming

$$\tilde{\mathbf{\Psi}} = [\tilde{\psi}_1 \tilde{\psi}_2 \cdots \tilde{\psi}_n \tilde{\psi}_1^* \tilde{\psi}_2^* \cdots \tilde{\psi}_n^*], \tag{19}$$

the frequency responses in state space, which are expressed in modal parameters in equation (18), can be expanded in complex modal space as

$$\mathbf{Y}(\omega) = \sum_{r=1}^{n} \frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{F}_S(\omega)}{\lambda_r - \mathrm{j}\omega}\tilde{\psi}_r + \sum_{r=1}^{n} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{F}_S(\omega)}{\lambda_r^* - \mathrm{j}\omega}\tilde{\psi}_r. \tag{20}$$

According to the definition of $\mathbf{Y}(\omega)$ and $\mathbf{F}_S(\omega)$ in equations (3) and (5), respectively, the frequency responses in physical space are

$$\mathbf{X}(\omega) = \sum_{r=1}^{n} \frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \sum_{r=1}^{n} \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*, \tag{21}$$

where, ψ_r and ψ_r^* are the rth vectors of matrices $\mathbf{\Psi}$ and $\mathbf{\Psi}^*$ respectively. They also are the upper-half parts of vectors $\tilde{\psi}_r$ and $\tilde{\psi}_r^*$, λ_r and λ_r^* are the rth diagonal elements of matrices $\mathbf{\Omega}$ and $\mathbf{\Omega}^*$ respectively.

The exciting frequencies can be classified into three categories compared with the natural frequencies of the system. (1) The excited frequencies are all very low and the largest one is still less than the lowest natural frequency of the system. For this case, the calculation of the frequency responses and their sensitivities is very simple and will not be discussed in the following. (2) The exciting frequencies are low and lie in the low-frequency range of the system. (3) The exciting frequencies are a little high and lie in the middle-frequency range. According to the division, the modal truncation can be divided into middle–high-modal and low–high-modal truncation schemes.

In the middle–high-modal truncation approach, the middle and high modes of the system are truncated. Hence, only the modes in the low-frequency range are applied to calculate the frequency responses and their sensitivities. Suppose that the low L groups of modes are selected when the modal superposition is applied, the frequency responses in state space are

$$\mathbf{X}_1^l(\omega) = \sum_{r=1}^{L} \frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \sum_{r=1}^{L} \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*. \tag{22}$$

When the exciting frequencies lie in the middle-frequency range of the system, the number of the kept modes will be very large if equation (22) is still used to calculate the frequency responses. This makes it difficult to solve the eigenproblem (12). Hence, the low–high-modal truncation approach is applied. If the L_1th through L_2th groups of modes are selected as the kept modes, the frequency responses can be expressed as

$$\mathbf{X}_1^m(\omega) = \sum_{r=L_1}^{L_2} \frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \sum_{r=L_1}^{L_2} \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*. \tag{23}$$

The modal truncation errors of frequency responses resulted from equations (22) and (23) are

$$\mathbf{E}_1^l(\omega) = \sum_{r=L+1}^{n} \frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \sum_{r=L+1}^{n} \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*, \tag{24}$$

$$\mathbf{E}_1^m(\omega) = \sum_{r=1}^{L_1-1} \left[\frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*\right] + \sum_{r=L_2+1}^{n} \left[\frac{\psi_r^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \frac{(\psi_r^*)^{\mathrm{T}}\mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*\right]. \tag{25}$$

The superscript l and m in equations (22)–(25) denote the frequency responses in the low- and middle-frequency range respectively. The subscript 1 in these equations denotes the results obtained from the modal superposition.

3.2. DOUBLE-MODAL SUPERPOSITION METHOD FOR THE SENSITIVITIES

Substituting equation (17) into equation (10) yields

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = \tilde{\mathbf{\Psi}}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{R}(\omega). \tag{26}$$

For the double-modal superposition method, three steps are required to calculate the sensitivities of frequency responses. (i) Compute the frequency responses of the system under external forces. (ii) Calculate the pseudo-force vector $\mathbf{R}(\omega)$ by using equation (8). (iii) After substituting $\mathbf{R}(\omega)$ into equation (26), the sensitivities are obtained by a second-modal superposition, i.e.,

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = \sum_{r=1}^{n} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r - \mathrm{j}\omega} \tilde{\psi}_r + \sum_{r=1}^{n} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \tilde{\psi}_r^*. \tag{27}$$

According to the definition of $\mathbf{Y}(\omega)$, $\tilde{\psi}_r$ and $\tilde{\psi}_r^*$, the sensitivities of frequency responses in physical space are

$$\frac{\partial \mathbf{X}(\omega)}{\partial p_j} = \sum_{r=1}^{n} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{n} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*. \tag{28}$$

When the middle–high-modal and the low–high-modal truncation schemes are applied, the sensitivities expressed in equation (28) can be written as

$$\begin{gathered} \left(\frac{\partial \mathbf{X}(\omega)}{\partial p_j}\right)_1^l = \sum_{r=1}^{L} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}_1^l(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{L} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}_1^l(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*. \\ \mathbf{R}_1^l(\omega) = \mathbf{S}(\omega)\mathbf{Y}_1^l(\omega), \qquad \mathbf{Y}_1^l(\omega) = \begin{Bmatrix} \mathbf{X}_1^l(\omega) \\ \mathrm{j}\omega \mathbf{X}_1^l(\omega) \end{Bmatrix}, \end{gathered} \tag{29}$$

$$\begin{gathered} \left(\frac{\partial \mathbf{X}(\omega)}{\partial p_j}\right)_1^m = \sum_{r=L_1}^{L_2} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}_1^m(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=L_1}^{L_2} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}_1^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*, \\ \mathbf{R}_1^m(\omega) = \mathbf{S}(\omega)\mathbf{Y}_1^m(\omega), \qquad \mathbf{Y}_1^m(\omega) = \begin{Bmatrix} \mathbf{X}_1^m(\omega) \\ \mathrm{j}\omega \mathbf{X}_1^m(\omega) \end{Bmatrix}, \end{gathered} \tag{30}$$

where $\mathbf{X}_1^l(\omega)$ and $\mathbf{X}_1^m(\omega)$ are defined by equations (22) and (23). The modal truncation errors of the sensitivities resulting from equations (29) and (30) are

$$\begin{aligned} \bar{\mathbf{E}}_1^l(\omega) = & \sum_{r=1}^{n} \left[\frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{S}(\omega) \tilde{\mathbf{E}}_1^l(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{S}(\omega) \tilde{\mathbf{E}}_1^l(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right] \\ & + \sum_{r=L-1}^{n} \left[\frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^l(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^l(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right], \end{aligned} \tag{31}$$

$$\begin{aligned} \bar{\mathbf{E}}_1^m(\omega) = & \sum_{r=1}^{n} \left[\frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{S}(\omega) \tilde{\mathbf{E}}_1^m(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{S}(\omega) \tilde{\mathbf{E}}_1^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right] \\ & + \sum_{r=1}^{L_1-1} \left[\frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^m(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right] \\ & + \sum_{r=L_2+1}^{n} \left[\frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^m(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{S}(\omega) \mathbf{Y}_1^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right], \end{aligned} \tag{32}$$

respectively, and

$$\tilde{\mathbf{E}}_1^l(\omega) = \begin{Bmatrix} \mathbf{E}_1^l(\omega) \\ \mathrm{j}\omega \mathbf{E}_1^l(\omega) \end{Bmatrix}, \qquad \tilde{\mathbf{E}}_1^m(\omega) = \begin{Bmatrix} \mathbf{E}_1^m(\omega) \\ \mathrm{j}\omega \mathbf{E}_1^m(\omega) \end{Bmatrix}. \tag{33}$$

Obviously, the errors of the sensitivities of frequency responses are composed of two parts. One results from the truncation errors of frequency responses, which is expressed by the first part of equations (31) and (32). The other results from the modal truncation when calculating the sensitivities, which is denoted by the residual part in the two equations. Usually, the modal truncation errors of the sensitivities are larger than the frequency responses because the modal truncation schemes are applied twice for the double-modal superposition method. Hence, it is necessary to introduce the modal acceleration method and double-modal acceleration method.

4. FREQUENCY RESPONSES AND THEIR SENSITIVITIES IN THE LOW-FREQUENCY RANGE

4.1. MODAL ACCELERATION METHOD FOR FREQUENCY RESPONSES

It can be proven that the inverse of matrix $(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})$ in equation (17) can be expanded in power series as [10]

$$(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1} = \tilde{\mathbf{\Omega}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{h} + (\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{H+1}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}, \tag{34}$$

where H is any integer that is larger than -1. $H = -1$ indicates that no power series is adopted. Substituting equation (34) into equation (18), the frequency responses in state space can be expressed as

$$\mathbf{Y}(\omega) = \mathbf{Y}_A(\omega) + \mathbf{Y}_S(\omega), \tag{35}$$

where $\mathbf{Y}_A(\omega)$ and $\mathbf{Y}_S(\omega)$ denote the frequency responses defined by the summation of the former $H+1$ items and by the residue of the power series respectively. They are

$$\mathbf{Y}_A(\omega) = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{h}\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{F}_S(\omega), \tag{36}$$

$$\mathbf{Y}_S(\omega) = \tilde{\mathbf{\Psi}}(\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{H+1}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{F}_S(\omega). \tag{37}$$

The frequency responses $\mathbf{Y}(\omega)$ are divided into two parts, $\mathbf{Y}_A(\omega)$ and $\mathbf{Y}_S(\omega)$, only because the two parts are associated with the modal acceleration and modal superposition respectively.

It can be seen from equation (36) that the frequency responses $\mathbf{Y}_A(\omega)$ are expressed using modal parameters of the system. However, almost all the high eigenvalues and eigenvectors are usually not available for a large and/or complex system. Hence, it is necessary to rewrite them by using physical parameters of the system. By considering equation (16), we have

$$\mathbf{A}^{-1}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^{h} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}} \underbrace{(\mathbf{B}\tilde{\mathbf{\Psi}}\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}) \cdots (\mathbf{B}\tilde{\mathbf{\Psi}}\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}})}_{h}. \tag{38}$$

Equation (38) can be simplified by using equation (14) as

$$\mathbf{A}^{-1}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^{h} = \tilde{\mathbf{\Psi}}\tilde{\mathbf{\Omega}}^{-1}(\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{h}\tilde{\mathbf{\Psi}}^{\mathrm{T}}. \tag{39}$$

Introducing equation (39) into equation (36) yields

$$\mathbf{Y}_A(\omega) = \mathbf{A}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^{h}\mathbf{F}_S(\omega). \tag{40}$$

Obviously, the parameters on the right-hand side of equation (40) are all known in advances.

The frequency responses $\mathbf{Y}_S(\omega)$ defined by the residue of the power series can be expanded in complex modal space as

$$\mathbf{Y}_S(\omega) = \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{F}_S(\omega)}{\lambda_r - \mathrm{j}\omega} \tilde{\psi}_r + \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{F}_S(\omega)}{\lambda_r^* - \mathrm{j}\omega} \tilde{\psi}_r^*. \tag{41}$$

Relatively, the frequency responses in physical space are

$$\mathbf{X}_S(\omega) = \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*. \tag{42}$$

Substituting equations (40) and (41) into the right-hand side of equation (35) and considering equations (3) and (42), the frequency responses in physical space are obtained as

$$\begin{aligned} \mathbf{X}(\omega) = &\left[\mathbf{A}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega \mathbf{B}\mathbf{A}^{-1})^h \mathbf{F}_S(\omega)\right]_U \\ &+ \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*, \end{aligned} \tag{43}$$

where subscript "U" denotes the upper part of the vector. Assuming the low L groups of modes are selected as the kept modes when the modal truncation is adopted, the frequency responses in physical space can be expressed as

$$\begin{aligned} \mathbf{X}_2^l(\omega) = &\left[\mathbf{A}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega \mathbf{B}\mathbf{A}^{-1})^h \mathbf{F}_S(\omega)\right]_U \\ &+ \sum_{r=1}^{L} \left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{L} \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*. \end{aligned} \tag{44}$$

The modal truncation errors resulting from equation (44) are

$$\mathbf{E}_2^l(\omega) = \sum_{r=L+1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=L+1}^{n} \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^*. \tag{45}$$

The subscript 2 in equations (44) and (45) denotes the results obtained from the modal acceleration method.

4.2. DOUBLE-MODAL ACCELERATION METHOD FOR THE SENSITIVITIES

After frequency responses are obtained, they can be used to calculate their sensitivities. Introducing equation (34) into equation (26), one has

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = \left(\frac{\partial \mathbf{Y}(\omega)}{\partial p_j}\right)_A + \left(\frac{\partial \mathbf{Y}(\omega)}{\partial p_j}\right)_S, \tag{46}$$

$$\left(\frac{\partial \mathbf{Y}(\omega)}{\partial p_j}\right)_A = \tilde{\mathbf{\Psi}} \tilde{\mathbf{\Omega}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\omega \tilde{\mathbf{\Omega}}^{-1})^h \tilde{\mathbf{\Psi}}^{\mathrm{T}} \mathbf{R}(\omega), \tag{47}$$

$$\left(\frac{\partial \mathbf{Y}(\omega)}{\partial p_j}\right)_S = \tilde{\mathbf{\Psi}}(\mathrm{j}\omega\tilde{\mathbf{\Omega}}^{-1})^{H+1}(\tilde{\mathbf{\Omega}} - \mathrm{j}\omega\mathbf{I})^{-1}\tilde{\mathbf{\Psi}}^{\mathrm{T}}\mathbf{R}(\omega), \tag{48}$$

Based on the same derivative procedure of equation (43), we have

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = \mathbf{A}^{-1}\sum_{h=0}^{H}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^h\mathbf{R}(\omega) + \sum_{r=1}^{n}\left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}(\omega)}{\lambda_r - \mathrm{j}\omega}\tilde{\psi}_r + \sum_{r=1}^{n}\left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{R}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\tilde{\psi}_r^*. \tag{49}$$

Equation (49) can be rewritten in physical space as

$$\frac{\partial \mathbf{X}(\omega)}{\partial p_j} = \left[\mathbf{A}^{-1}\sum_{k=0}^{H}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^h\mathbf{R}(\omega)\right]_U + \sum_{r=1}^{n}\left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r$$

$$+ \sum_{r=1}^{n}\left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1}\frac{(\psi_r^*)^{\mathrm{T}}\mathbf{R}(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*. \tag{50}$$

When the low L groups of modes are chosen as the kept modes, the sensitivities of frequency responses in physical space are obtained by the double-modal acceleration method as

$$\left(\frac{\partial \mathbf{X}(\omega)}{\partial p_j}\right)_2^l = \left[\mathbf{A}^{-1}\sum_{h=0}^{H}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^h\mathbf{R}_2^l(\omega)\right]_U + \sum_{r=1}^{L}\left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}_2^l(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r$$

$$+ \sum_{r=1}^{L}\left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{R}_r^l(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*, \tag{51}$$

$$\mathbf{R}_2^l(\omega) = \mathbf{S}(\omega)\mathbf{Y}_2^l(\omega), \qquad \mathbf{Y}_2^l(\omega) = \begin{Bmatrix} \mathbf{X}_2^l(\omega) \\ \mathrm{j}\omega\mathbf{X}_2^l(\omega) \end{Bmatrix}.$$

Similarly, three steps are required to calculate the sensitivities of frequency responses by using the double-modal acceleration method. (i) The frequency response of the system under exciting forces are computed by using the modal acceleration method in equation (44). (ii) Pseudo-force vector $\mathbf{R}_2^l(\omega)$ is calculated by using the second equation of equation (51). (iii) $\mathbf{R}_2^l(\omega)$ is substituted into the first equation of equation (51), and then the sensitivities are obtained by a second-modal acceleration.

The truncation errors resulting from equation (51) are

$$\bar{\mathbf{E}}_2^l(\omega) = \left[\mathbf{A}^{-1}\sum_{h=0}^{H}(\mathrm{j}\omega\mathbf{B}\mathbf{A}^{-1})^h\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^l(\omega)\right]_U$$

$$+ \sum_{r=1}^{n}\left[\left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^l(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^l(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*\right] \tag{52}$$

$$+ \sum_{r=L+1}^{n}\left[\left(\frac{\mathrm{j}\omega}{\lambda_r}\right)^{H-1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}_2^l(\omega)}{\lambda_r - \mathrm{j}\omega}\psi_r + \left(\frac{\mathrm{j}\omega}{\lambda_r^*}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{R}_2^l(\omega)}{\lambda_r^* - \mathrm{j}\omega}\psi_r^*\right],$$

where

$$\tilde{\mathbf{E}}_2^l(\omega) = \begin{Bmatrix} \mathbf{E}_2^l(\omega) \\ \mathrm{j}\omega\mathbf{E}_2^l(\omega) \end{Bmatrix}. \tag{53}$$

The errors of the sensitivities are composed of two parts. This is similar to the structure of equation (31). One results from the truncation errors of frequency responses calculated using equation (44) and is expressed by the former two parts of equation (52). The other results from modal truncation when calculating the sensitivities themselves and is denoted by the third part of equation (52).

Equation (45) has a coefficient compared with equation (24). In order that the truncation errors of frequency responses obtained from equation (44) reduce with the increase of the items H of the power series, the inequalities

$$\left|\frac{\mathrm{j}\omega}{\lambda_r}\right| < 1, \qquad \left|\frac{\mathrm{j}\omega}{\lambda_r^*}\right| < 1, \quad (r > L) \tag{54}$$

should be satisfied for the truncated modes. Considering

$$\lambda_r = -\xi_r\omega_r + \mathrm{j}\omega_r\sqrt{1-\xi_r^2}, \quad \lambda_r^* = -\xi_r\omega_r - \mathrm{j}\omega_r\sqrt{1-\xi_r^2} \tag{55}$$

one has

$$\omega_{max}^2 < \omega_{L+1}^2, \tag{56}$$

where ξ_r and ω_r are the rth modal damping and frequency of the system. ω_{max} is the highest frequency of the exciting frequencies. Usually, one or two more modes are selected to make the convergence faster. Obviously, when condition (54) or (56) is satisfied, the modal acceleration method can make the approximate frequency responses $\mathbf{X}_2^l(\omega)$ very close to the exact $\mathbf{X}(\omega)$, which leads to $|\mathbf{E}_2^l(\omega)| \ll |\mathbf{X}_2^l(\omega)|$. Similarly, when condition (54) is satisfied the errors of the sensitivities of frequency responses are convergent too. Hence, inequality (54) or (56) is the convergent condition of the frequency responses and their sensitivities.

5. FREQUENCY RESPONSES AND THEIR SENSITIVITIES IN THE MIDDLE-FREQUENCY RANGE

5.1. MODEL ACCELERATION METHOD FOR FREQUENCY RESPONSES

Considering the eigenvalue shifting technique, we have

$$(\mathbf{A} - \mathrm{j}\omega\mathbf{B}) = (\bar{\mathbf{A}} - \mathrm{j}\bar{\omega}\mathbf{B}), \tag{57}$$

where

$$\bar{\mathbf{A}} = \mathbf{A} - \mathrm{j}q\mathbf{B}, \qquad \bar{\omega} = \omega - q. \tag{58}$$

Usually, the eigenvalue shifting q is

$$q = \frac{\omega_{max} + \omega_{min}}{2}. \tag{59}$$

ω_{min} and ω_{max} are the under and upper boundary values of the exciting frequencies. Substituting equation (57) into equation (6), the frequency responses become

$$\mathbf{Y}(\omega) = (\bar{\mathbf{A}} - \mathrm{j}\omega\bar{\mathbf{B}})^{-1}\mathbf{F}_S(\omega). \tag{60}$$

When the modal acceleration method is applied, the frequency responses can be expressed as

$$\mathbf{X}(\omega) = \left[\bar{\mathbf{A}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\bar{\omega}\mathbf{B}\bar{\mathbf{A}}^{-1})^h \mathbf{F}_S(\omega) \right]_U + \sum_{r=1}^{n} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{n} \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathbf{j}\omega} \psi_r^* \right]. \tag{61}$$

Suppose that the L_1th through L_2th modes are selected as the kept modes when the modal truncation is applied, the frequency responses in the middle-frequency range are

$$\mathbf{X}_2^m(\omega) = \left[\bar{\mathbf{A}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\bar{\omega}\mathbf{B}\bar{\mathbf{A}}^{-1})^h \mathbf{F}_S(\omega) \right]_U + \sum_{r=L_1}^{L_2} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right]. \tag{62}$$

The errors resulting from equation (62) are

$$\mathbf{E}_2^m(\omega) = \sum_{r=1}^{L_1 - 1} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right] + \sum_{r=L_2+1}^{n} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\psi_r^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\psi_r^*)^{\mathrm{T}} \mathbf{F}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right]. \tag{63}$$

5.2. DOUBLE-MODAL ACCELERATION METHOD FOR THE SENSITIVITIES

Introducing equation (57) into equation (10) yields

$$\frac{\partial \mathbf{Y}(\omega)}{\partial p_j} = (\bar{\mathbf{A}} - \mathrm{j}\bar{\omega}\mathbf{B})^{-1} \mathbf{R}(\omega). \tag{64}$$

Based on the same derivation above, one has

$$\frac{\partial \mathbf{X}(\omega)}{\partial p_j} = \left[\bar{\mathbf{A}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\bar{\omega}\mathbf{B}\bar{\mathbf{A}}^{-1})^h \mathbf{R}(\omega) \right]_U + \sum_{r=1}^{n} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r - \mathrm{j}\omega} \psi_r + \sum_{r=1}^{n} \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* . \tag{65}$$

When the modal truncation is applied, the sensitivities are

$$\left(\frac{\partial \mathbf{X}(\omega)}{\partial p_j} \right)_2^m = \left[\bar{\mathbf{A}}^{-1} \sum_{h=0}^{H} (\mathrm{j}\bar{\omega}\mathbf{B}\bar{\mathbf{A}}^{-1})^h \mathbf{R}_2^m(\omega) \right]_U + \sum_{r=L_1}^{L_2} \left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r - \mathrm{j}q} \right)^{H+1} \frac{\tilde{\psi}_r^{\mathrm{T}} \mathbf{R}_2^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r + \left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^* - \mathrm{j}q} \right)^{H+1} \frac{(\tilde{\psi}_r^*)^{\mathrm{T}} \mathbf{R}_2^m(\omega)}{\lambda_r^* - \mathrm{j}\omega} \psi_r^* \right], \tag{66}$$

$$\mathbf{R}_2^m(\omega) = \mathbf{S}(\omega)\mathbf{Y}_2^m(\omega), \qquad \mathbf{Y}_2^m(\omega) = \begin{Bmatrix} \mathbf{X}_2^m(\omega) \\ \mathrm{j}\omega\mathbf{X}_2^m(\omega) \end{Bmatrix}.$$

The errors resulting from equation (66) are

$$\bar{\mathbf{E}}_2^m = \left[\bar{\mathbf{A}}^{-1}\sum_{h=0}^{H}(\mathrm{j}\bar{\omega}\mathbf{B}\bar{\mathbf{A}}^{-1})^h\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^m(\omega)\right]_U$$
$$+\sum_{r=1}^{n}\left[\left(\frac{\mathrm{j}\omega}{\lambda_r-\mathrm{j}q}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^m}{\lambda_r-\mathrm{j}\omega}\psi_r+\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^*-\mathrm{j}q}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{S}(\omega)\tilde{\mathbf{E}}_2^m(\omega)}{\lambda_r^*-\mathrm{j}\omega}\psi_r^*\right]$$
$$+\sum_{r=1}^{L_1-1}\left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r-\mathrm{j}q}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}_2^m(\omega)}{\lambda_r-\mathrm{j}\omega}\psi_r+\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^*-\mathrm{j}q}\right)^{H+1}\frac{(\tilde{\psi}_r^*)^{\mathrm{T}}\mathbf{R}_2^m(\omega)}{\lambda_r^*-\mathrm{j}\omega}\mathbf{\Psi}_r\right]$$
$$+\sum_{r=L_2+1}^{n}\left[\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r-\mathrm{j}q}\right)^{H+1}\frac{\tilde{\psi}_r^{\mathrm{T}}\mathbf{R}_2^m(\omega)}{\lambda_r-\mathrm{j}\omega}\psi_r+\left(\frac{\mathrm{j}\bar{\omega}}{\lambda_r^*-\mathrm{j}q}\right)^{H+1}\frac{(\tilde{\psi}_r)^{\mathrm{T}}\mathbf{R}_2^m(\omega)}{\lambda_r^*-\mathrm{j}\omega}\mathbf{\Psi}_r\right], \quad (67)$$

where

$$\tilde{\mathbf{E}}_2^m(\omega) = \left\{\begin{matrix}\mathbf{E}_2^m(\omega)\\ \mathrm{j}\omega\mathbf{E}_2^m(\omega)\end{matrix}\right\}. \quad (68)$$

In order that the frequency responses and their sensitivities obtained from the modal acceleration method and the double-modal acceleration method are more accurate than those from the typical modal superposition method and that the truncation errors decrease with the increase of the items H of the power series, the selected L_1 and L_2 should satisfy

$$\left|\frac{\mathrm{j}(\omega-q)}{\lambda_r-jq}\right|<1, \qquad \left|\frac{\mathrm{j}(\omega-q)}{\lambda_r^*-\mathrm{j}q}\right|<1 \quad (r\leqslant L_1-1, \quad r\geqslant L_2+1) \quad (69)$$

that is

$$\omega_{L_2+1} > q\sqrt{1-\xi_r^2}+\sqrt{q^2(1-\xi_r^2)-2q\omega_{max}+\omega_{max}^2},$$
$$\omega_{L_2-1} < q\sqrt{1-\xi_r^2}-\sqrt{q^2(1-\xi_r^2)-2q\omega_{max}+\omega_{max}^2}. \quad (70)$$

When the damping ratio is small, the following inequalities can be derived from inequalities (70) approximately:

$$\omega_{L_2+1} > \omega_{max}, \qquad \omega_{L_1-1} < \omega_{min}. \quad (71)$$

Inequalities (69) or (71) are the governing inequalities of L_1 and L_2. It means that the frequencies corresponding to the truncated modes should lie outside of the exciting frequency range. Usually, one or two more modes are selected to make the convergence faster. When inequalities (69) or (71) are satisfied, the truncation errors of the sensitivities decrease with the increase of H.

6. NUMERICAL EXAMPLE

Floating raft isolation system, which has been developed during the past 20 years, is an efficient equipment for vibration isolation and noise reduction [10]. It can effectively isolate the vibrations of the host and auxiliary machines and reduce the structural noise of ships and submarines. Floating raft isolation system will protect the equipment in ships or

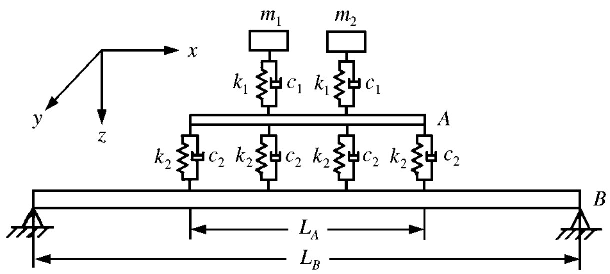

Figure 1. Schematic of floating raft isolation system.

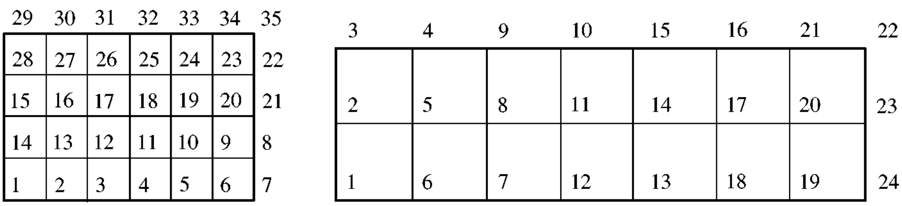

Figure 2. (a) Finite element model of the raft frame; (b) finite element model of the base.

submarines from damage and let them work normally when the ships or submarines are subjected to external loads or shocks. Floating raft isolation system is a compound dynamic system. It contains springs, dampers, machines to be isolated, raft frame and base.

The schematic of a floating raft isolation system is shown in Figure 1: $m_1 = 100$ kg and $m_2 = 120$ kg denote the machines which are to be isolated. A and B are rectangular plates and denote the raft frame and base respectively. The length–width–thickness of them are 12 m–0·8 m–0.02 m and 2·8 m–0·8 m–0·04 m respectively. Their modulus of elasticity = 2·0E11 N/m^2, mass density = 7800 kg/m^3. The two short sides of plate B are simply supported and two long sides are free. The four sides of plate A are all free. $k_1 = 1{\cdot}0\text{E}5$ N/m, $k_2 = 5{\cdot}0\text{E}5$ N/m, $c_1 = 200$ n s/m^2, $c = 400$ n s/m^2.

The finite element model of the raft frame is shown in Figure 2(a). The model has 24 rectangular elements, 35 nodes and 105 d.o.f. The nodes which are connected with spring and damper I, k_1 and c_1, are 17 and 19. The nodes those are connected with spring and damper II are 1, 3, 5, 7, 15, 17, 19, 21, 29, 31, 33 and 35. The finite element model of the base is shown in Figure 2(b). The model has 14 rectangular elements, 24 nodes and 72 d.o.f. The nodes those are connected with spring and damper II are 7, 12, 13, 18, 8, 11, 14, 17, 9, 10, 15 and 16. The isolation system has a total of 179 d.o.f. The former 48 complex frequencies of the full model are listed in Table 1.

6.1. FREQUENCY RESPONSES

Assume that an identity force is acted on m_1 in the z direction. The frequency range of the force is 0–500 rad/s. It lies in the low-frequency range of the structure. According to the frequencies of the exciting forces and equation (56), $L = 9$ is selected. This means that all the modes which are higher than the 9th are truncated when the middle-high-modal truncation scheme is adopted. The frequency responses at node 10 of the base in the z direction for various H are calculated and drawn in Figure 3(a). For convenience, only the amplitudes are considered here and in the following. The errors of these approximate frequency responses

Table 1

Former 48 complex frequencies of the system in Figure 1 (rad/s)

Mode	Frequency	Mode	Frequency	Mode	Frequency	Mode	Frequency
1	−0·668763 ± j27·4761	13	−50·3217 ± j1052·66	25	−6·43327 ± j2571·18	37	−33·9316 ± j4292·75
2	−0·908139 ± j30·7180	14	−51·6692 ± j1124·83	26	−37·2371 ± j2599·01	38	−93·2515 ± j4367·52
3	−0·643143 ± j67·4590	15	−2·44333 ± j1244·83	27	−5·28205 ± j2831·25	39	−1·67897 ± j4370·59
4	−19·5565 ± j227·402	16	−6·81218 ± j1264·11	28	−19·5478 ± j2917·43	40	−48·7835 ± j4686·78
5	−13·0252 ± j227·648	17	−39·9179 ± j1278·33	29	−8·61524 ± j2961·48	41	−0·000000 ± j4875·89
6	−13·5133 ± j239·506	18	−72·0624 ± j1482·56	30	−57·5689 ± j3116·79	42	−105·467 ± j4944·46
7	−14·9352 ± j334·254	19	−5·11117 ± j1858·42	31	−116·931 ± j3118·04	43	−6·37971 ± j5187·60
8	−25·8998 ± j416·102	20	−66·5287 ± j1869·18	32	−47·0278 ± j3204·45	44	−12·2187 ± j5503·62
9	−36·0185 ± j534·619	21	−3·54278 ± j1956·76	33	−9·59058 ± j3406·84	45	−37·8496 ± j5798·07
10	−32·9596 ± j542·768	22	−67·7733 ± j2120·51	34	−8·16363 ± j3586·46	46	−2·74991 ± j5838·25
11	−4·52826 ± j697·487	23	−17·9071 ± j2245·33	35	−67·1672 ± j3887·99	47	−42·2488 ± j5883·27
12	−12·0737 ± j782·115	24	−12·1321 ± j2513·00	36	−0·00000 ± j4196·49	48	−110·980 ± j5980·82

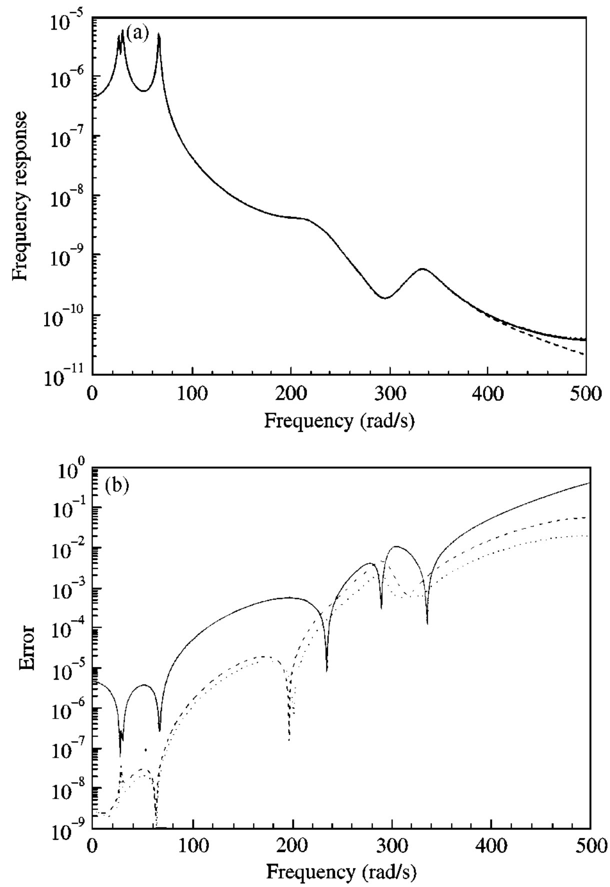

Figure 3. Frequency responses and their errors in the low-frequency range for different H's. (a) Frequency responses for different H's: —— exact; - - - - - $H = -1$; · · · · · · $H = 0$; –·–·– ·– $H = 2$; (b) Errors of the frequency responses for different H's: —— $H = -1$; - - - - - $H = 0$; · · · · · · $H = 2$.

are shown in Figure 3(b). The *Error* is defined as

$$Error = |(x_{appro} - x_{exact})/x_{exact}| \tag{72}$$

where x_{appro} and x_{exact} denote the approximate and exact frequency responses or sensitivities respectively. In the two figures. $H = -1$ denotes the frequency responses obtained from equation (22).

The accuracy of frequency responses obtained from equation (22) is low especially when the exciting frequency closes to 500 rad/s. The largest percent error (= 100Error%), for example, is 44·85%. After the modal acceleration method is applied, the accuracy increases quickly. For the cases of $H = 0$ and 2, the largest percent errors are 5·48 and 2·04% respectively. The accuracy of the frequency responses for the lower frequencies increases more quickly than those for the higher frequencies. This can be seen from Figure 3(b).

Suppose the frequency range of the exciting forces is 4000–5000 rad/s. It lies in the middle-frequency range of the isolation system. According to the frequencies of the exciting forces and equation (59), $q = 4500$, $L_1 = 35$ and $L_2 = 43$ are selected. This means that the former 34 modes and all the modes which are higher than the 43rd are truncated when the

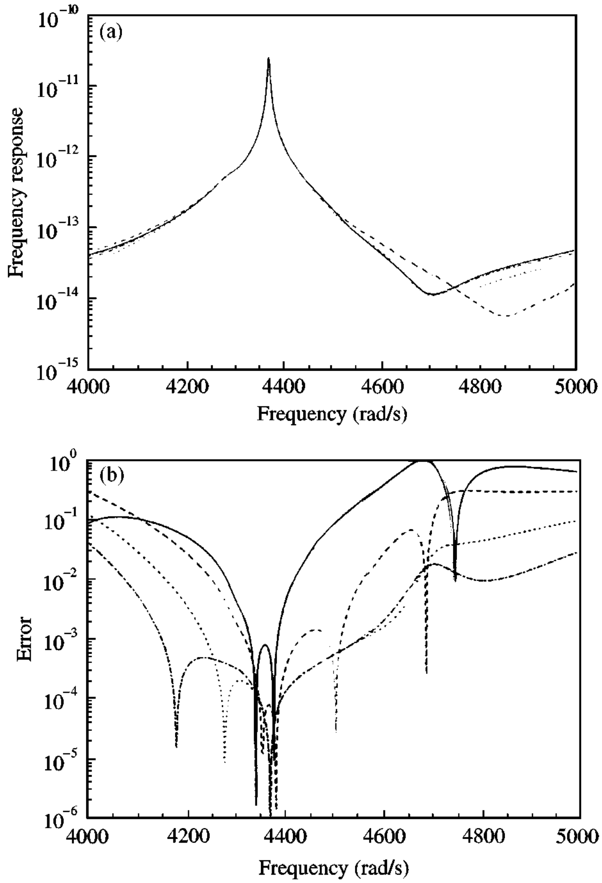

Figure 4. Frequency responses and their errors in the middle-frequency range for different H's. (a) Frequency responses for different H's: —— exact; - - - - - $H = -1$; · · · · · · $H = 0$; – · – · – · – $H = 2$; – · · – · · – $H = 4$. (b) Errors of the frequency responses for different H's: —— $H = -1$; - - - - - $H = 0$; · · · · · · $H = 2$; – · – · – · – $H = 4$.

low-high-modal truncation scheme is adopted. The frequency responses at m_1 in the z direction for various H are calculated and shown in Figures (4a). The errors of these approximate frequency responses are shown in Figure 4(b). In the two figures, $H = -1$ denotes the approximate frequency responses obtained from equation (23).

The accuracy of frequency responses obtained from equation (23) is very low. The largest percent error, for example, is 103·2%. Obviously, these frequency responses are useless. After the modal acceleration method is applied, the accuracy increases very quickly. The percent errors at $\omega = 5000$ rad/s for $H = -1$, 0, 2 and 4 are 65·14, 30·54, 9·97 and 3·01% respectively. The accuracy of the frequency responses for middle frequencies increases much more quickly than that for the lower and higher frequencies.

6.2. SENSITIVITIES OF FREQUENCY RESPONSES

Assume that the stiffness of the spring I under m_1 is selected as the design parameter. The sensitivities of the frequency responses discussed above are calculated and shown in Figures 5(a) and 6(a). Their errors are shown in Figures 5(b) and 6(b). The accuracy of the

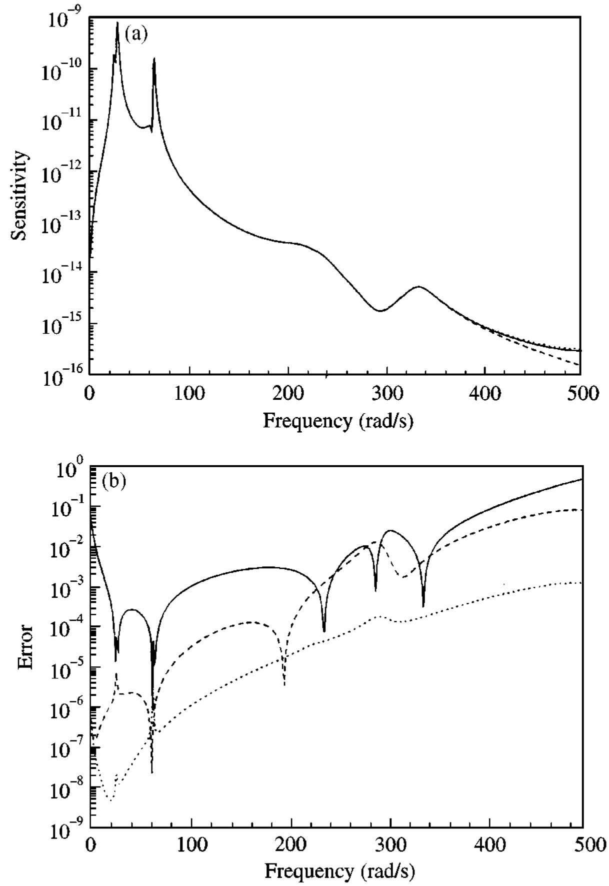

Figure 5. Sensitivities of frequency responses and their errors in the low-frequency range for different H's. (a) Sensitivities of frequency responses for different H's: —— exact; - - - - - $H = -1$; · · · · · · $H = 0$; –·–·– ·– $H = 2$. (b) Errors of the sensitivities of frequency responses for different H's: —— $H = -1$; - - - - - $H = 0$; · · · · · · $H = 2$.

sensitivities obtained from the double-modal superposition method, i.e., equations (29) and (30), is very low. It increases quickly if the double-modal acceleration method is applied. When the former three items of the power series, that is $H = 2$, are adopted, the errors are reduced by hundreds of times. The errors for $H = -1$ in Figure 6(b) are almost equal to 1 because the absolute values of the sensitivities obtained from the modal superposition method are much smaller than the exact, which can be seen clearly from Figure 6(a).

Comparing Figures 3(b) with 4(b), 5(b) with 6(b), one can find that the errors of frequency responses and their sensitivities in the low-frequency range are much smaller than those in the middle-frequency range. This means the modal acceleration method is much more necessary for the frequency responses and their sensitivities in the middle-frequency range than in the low-frequency range. For example, the sensitivities obtained from the double-modal superposition method shown in Figure 6(a) are much smaller than the exact. When the modal acceleration method is applied, the accuracy increases very quickly.

It can be seen from Figures 3(b) and 5(b), 4(b) and 6(b) that if the same modes are selected to calculate the frequency responses and their sensitivities, the accuracy of the former is usually higher than the latter. This is because the modal truncation scheme is used once in

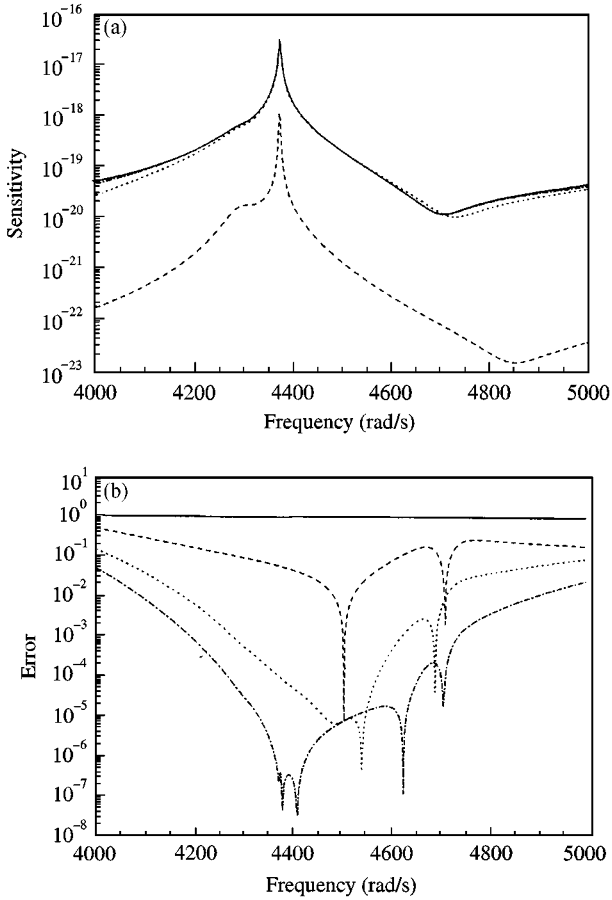

Figure 6. Sensitivities of frequency responses and their errors in the middle-frequency range for different H's. (a) Sensitivities of frequency responses for different H's: —— exact; - - - - - $H = -1$; · · · · · · $H = 0$; –·–·– ·– $H = 2$; –··–··– $H = 4$. (b) Errors of the sensitivities of frequency responses for different H's: —— $H = -1$; - - - - - $H = 0$; · · · · · · $H = 2$; –·–·– ·– $H = 4$.

the calculation of the former while it is twice for the latter. Hence, modal acceleration method is much more useful for the sensitivity than for the frequency response.

One also can see from Figures 3(a), 4(a), 5(a), and 6(a) that only several items of the series, such as 2 or 3, can make the frequency response and their sensitivities close to the exact accurately.

7. CONCLUSIONS

Based on the hybrid expansion of the inverse of the complex dynamic system matrix, one modal acceleration method for frequency responses and one double-modal acceleration method for their sensitivities of viscously damped systems are derived respectively. Two-modal truncation schemes are proposed. When the frequencies of exciting forces lie in the low-frequency range of the system, the middle and high modes can be truncated by using the present methods. When the exciting frequencies lie in the middle-frequency range, the high as well as the low modes can be truncated at the same time. The following conclusions can be drawn from these methods:

(1) Theoretically, the natural frequencies corresponding to the truncated modes should lie outside of the frequency range of the exciting forces when using the proposed methods. However, one or two more modes are selected as the kept modes to improve the convergence of the acceleration methods.
(2) The accuracy of frequency responses and their sensitivities obtained from the modal superposition method and the double-modal superposition method are very low. When two-modal acceleration methods are adopted, the modal truncation errors reduce very quickly.
(3) If the same modes are adopted for calculating frequency responses and their sensitivities, the accuracy of the former is usually higher than the latter because one more modal truncation is applied for the latter. Hence, modal acceleration approach is much more necessary for the latter than for the former.
(4) The modal truncation errors of the frequency responses and their sensitivities in the low-frequency range are smaller than those in middle-frequency range when modal truncation is applied. Hence, modal acceleration method is much more necessary for the frequency responses in the middle-frequency range.
(5) Generally, highly accurate results can be obtained when several items of the power series are adopted. Hence these methods are very efficient for frequency responses and their sensitivities, especially for the latter.
(6) The proposed methods are also valid for frequency response functions, responses in time domain and their sensitivities.
(7) If one uses subspace iteration method to solve the complex eigenproblem in equation (12) [11], the decomposition of matrix $\mathbf{A}$ or $\bar{\mathbf{A}}$ is required in this method. Hence it is unnecessary to decompose it again for the present methods. This makes the proposed acceleration methods much more computationally efficient.

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APPENDIX: NOMENCLATURE

$\mathbf{0}$	$(n \times n)$ null matrix or zero vector
$\mathbf{A} =$	$(2n \times 2n)$ system matrix in state space
$\bar{\mathbf{A}}$	$(2n \times 2n)$ modified system matrix in state space defined in equation (58)
$\mathbf{B}$	$(2n \times 2n)$ system matrix in state space
$\mathbf{C}$	$(n \times n)$ damping matrix in physical space
$\mathbf{E}$	(n) modal truncation error vector of frequency responses in physical space
$\tilde{\mathbf{E}}$	$(2n)$ modal truncation error vector of frequency responses in state space
$\bar{\mathbf{E}}$	(n) modal truncation error vector of the sensitivities of frequency responses in physical space
$\mathbf{F}$	(n) external force vector in physical space
$\mathbf{F}_S$	$(2n)$ external force vector in state space
$\mathbf{I}$	$(2n \times 2n)$ identity matrix
j	$\sqrt{-1}$
$\mathbf{K}$	$(n \times n)$ stiffness matrix in physical space
$\mathbf{M}$	$(n \times n)$ mass matrix in physical space
p	(m) design parameter vector
q	eigenvalue shifting
$\mathbf{R}$	$(2n)$ pseudo-force vector in state space defined in equation (8)
$\mathbf{S}$	$(2n \times 2n)$ sensitivity matrix in state space defined in equation (9)
$\mathbf{X}$	(n) frequency response vector in physical space
$\mathbf{Y}$	$(2n)$ frequency response vector in state space
λ	complex frequency
ω	exciting frequency
$\bar{\omega}$	exciting frequency with eigenvalue shifting defined in equation (58)
$\tilde{\mathbf{\Omega}}$	$(2n \times 2n)$ complex eigenvalue matrix in state space
$\mathbf{\Omega}$	$(n \times n)$ submatrix of the complex eigenvalue matrix defined in equation (11)
$\tilde{\mathbf{\Psi}}$	$(2n \times 2n)$ complex eigenvector matrix in state space
$\mathbf{\Psi}$	$(n \times n)$ submatrix of the complex eigenvector matrix defined in equation (11)
$\tilde{\psi}_i$	$(2n)$ the ith eigenvector
ψ_i	(n) up-half part of the ith eigenvector

Superscript

l	in the low-frequency range
m	in the middle-frequency range
T	matrix transpose
*	matrix complex conjugation

Subscript

r	rth mode